X100/301

NATIONAL QUALIFICATIONS 2005

FRIDAY, 20 MAY
9.00 AM - 10.10 AM

MATHEMATICS
HIGHER
Units 1, 2 and 3
Paper 1
(Non-calculator)

Read Carefully

1 Calculators may NOT be used in this paper.
2 Full credit will be given only where the solution contains appropriate working.
3 Answers obtained by readings from scale drawings will not receive any credit.
FORMULAE LIST

Circle:
The equation \( x^2 + y^2 + 2gx + 2fy + c = 0 \) represents a circle centre \((g, f)\) and radius \(\sqrt{g^2 + f^2 + c}\).
The equation \((x - a)^2 + (y - b)^2 = r^2\) represents a circle centre \((a, b)\) and radius \(r\).

Scalar Product:
\[ a \cdot b = |a| \cdot |b| \cdot \cos \theta, \text{ where } \theta \text{ is the angle between } a \text{ and } b \]
or
\[ a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 \text{ where } a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \]

Trigonometric formulae:
\[ \sin (A \pm B) = \sin A \cos B \pm \cos A \sin B \]
\[ \cos (A \pm B) = \cos A \cos B \mp \sin A \sin B \]
\[ \sin 2A = 2\sin A \cos A \]
\[ \cos 2A = \cos^2 A - \sin^2 A \]
\[ = 2\cos^2 A - 1 \]
\[ = 1 - 2\sin^2 A \]

Table of standard derivatives:

<table>
<thead>
<tr>
<th>( f(x) )</th>
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ALL questions should be attempted. 

1. Find the equation of the line ST, where T is the point (-2, 0) and angle STO is 60°.

2. Two congruent circles, with centres A and B, touch at P.
Relative to suitable axes, their equations are
\( x^2 + y^2 + 6x + 4y - 12 = 0 \) and
\( x^2 + y^2 - 6x - 12y + 20 = 0 \).
(a) Find the coordinates of P.
(b) Find the length of AB.

3. D,OABC is a pyramid. A is the point (12, 0, 0), B is (12, 6, 0) and D is (6, 3, 9).
F divides DB in the ratio 2:1.
(a) Find the coordinates of the point F.
(b) Express \( \vec{AF} \) in component form.
4. Functions \( f(x) = 3x - 1 \) and \( g(x) = x^2 + 7 \) are defined on the set of real numbers.
   
   (a) Find \( h(x) \) where \( h(x) = g(f(x)) \).
   
   (b) (i) Write down the coordinates of the minimum turning point of \( y = h(x) \).
   
   (ii) Hence state the range of the function \( h \).

5. Differentiate \((1 + 2 \sin x)^4\) with respect to \( x \).

6. (a) The terms of a sequence satisfy \( u_{n+1} = ku_n + 3 \). Find the value of \( k \) which produces a sequence with a limit of 4.
   
   (b) A sequence satisfies the recurrence relation \( u_{n+1} = mu_n + 5 \), \( u_0 = 3 \).
   
   (i) Express \( u_1 \) and \( u_2 \) in terms of \( m \).
   
   (ii) Given that \( u_2 = 7 \), find the value of \( m \) which produces a sequence with no limit.

7. The function \( f \) is of the form \( f(x) = \log_b (x - a) \).
   
   The graph of \( y = f(x) \) is shown in the diagram.
   
   (a) Write down the values of \( a \) and \( b \).
   
   (b) State the domain of \( f \).

8. A function \( f \) is defined by the formula \( f(x) = 2x^3 - 7x^2 + 9 \) where \( x \) is a real number.
   
   (a) Show that \((x - 3)\) is a factor of \( f(x) \), and hence factorise \( f(x) \) fully.
   
   (b) Find the coordinates of the points where the curve with equation \( y = f(x) \) crosses the x- and y-axes.
   
   (c) Find the greatest and least values of \( f \) in the interval \(-2 \leq x \leq 2 \).

9. If \( \cos 2x = \frac{7}{25} \) and \( 0 < x < \frac{\pi}{2} \), find the exact values of \( \cos x \) and \( \sin x \).
10. (a) Express \( \sin x - \sqrt{3} \cos x \) in the form \( k \sin (x - a) \) where \( k > 0 \) and \( 0 \leq a \leq 2\pi \).

(b) Hence, or otherwise, sketch the curve with equation \( y = 3 + \sin x - \sqrt{3} \cos x \) in the interval \( 0 \leq x \leq 2\pi \).

11. (a) A circle has centre \((t, 0)\), \( t > 0 \), and radius 2 units.
   Write down the equation of the circle.

(b) Find the exact value of \( t \) such that the line \( y = 2x \) is a tangent to the circle.

[END OF QUESTION PAPER]
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or \( \mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \) where \(\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \) and \(\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \).

Trigonometric formulae:
\[
\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B \\
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1. Find \( \int \frac{4x^4 - 1}{x^2} \, dx \), \( x \neq 0 \).

2. Triangles ACD and BCD are right-angled at D with angles \( p \) and \( q \) and lengths as shown in the diagram.
   
   (a) Show that the exact value of \( \sin(p + q) \) is \( \frac{84}{85} \).
   
   (b) Calculate the exact values of:
       
       (i) \( \cos(p + q) \);
       
       (ii) \( \tan(p + q) \).

3. (a) A chord joins the points A(1,0) and B(5,4) on the circle as shown in the diagram. Show that the equation of the perpendicular bisector of chord AB is \( x + y = 5 \).

   (b) The point C is the centre of this circle. The tangent at the point A on the circle has equation \( x + 3y = 1 \).
   
   Find the equation of the radius CA.

   (c) (i) Determine the coordinates of the point C.
   
   (ii) Find the equation of the circle.

   [Turn over]
4. The sketch shows the positions of Andrew(A), Bob(B) and Tracy(T) on three hill-tops. 
Relative to a suitable origin, the coordinates (in hundreds of metres) of the three people are A(23, 0, 8), B(-12, 0, 9) and T(28, -15, 7).
In the dark, Andrew and Bob locate Tracy using heat-seeking beams.
(a) Express the vectors $\vec{TA}$ and $\vec{TB}$ in component form.
(b) Calculate the angle between these two beams.

5. The curves with equations $y = x^2$ and $y = 2x^2 - 9$ intersect at K and L as shown.
Calculate the area enclosed between the curves.

6. The diagram shows the graph of $y = \frac{24}{\sqrt{x}}$, $x > 0$.
Find the equation of the tangent at P, where $x = 4$.

7. Solve the equation $\log_4(5 - x) - \log_4(3 - x) = 2$, $x < 3$. 

Marks

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<td>4(a)</td>
<td>2</td>
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<td>4(b)</td>
<td>5</td>
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<tr>
<td>5</td>
<td>8</td>
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<td>6</td>
<td>6</td>
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<td>7</td>
<td>4</td>
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8. Two functions, \( f \) and \( g \), are defined by \( f(x) = k \sin 2x \) and \( g(x) = \sin x \) where \( k > 1 \).

The diagram shows the graphs of \( y = f(x) \) and \( y = g(x) \) intersecting at \( O, A, B, C \) and \( D \).

Show that, at \( A \) and \( C \), \( \cos x = \frac{1}{2k} \).

9. The value \( V \) (in £ million) of a cruise ship \( t \) years after launch is given by the formula \( V = 252e^{-0.0633t} \).

(a) What was its value when launched?

(b) The owners decide to sell the ship once its value falls below £20 million. After how many years will it be sold?

10. Vectors \( \mathbf{a} \) and \( \mathbf{c} \) are represented by two sides of an equilateral triangle with sides of length 3 units, as shown in the diagram.

Vector \( \mathbf{b} \) is 2 units long and \( \mathbf{b} \) is perpendicular to both \( \mathbf{a} \) and \( \mathbf{c} \).

Evaluate the scalar product \( \mathbf{a} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \).

11. (a) Show that \( x = -1 \) is a solution of the cubic equation \( x^3 + px^2 + px + 1 = 0 \).

(b) Hence find the range of values of \( p \) for which all the roots of the cubic equation are real.

[END OF QUESTION PAPER]