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2002

Worked Solutions

With courtesy to the SQA and author(s)

2002 – Higher Paper I

1) Centre = $(-1, 1)$

$M_{radius} = \frac{1-3}{-1-2} = \frac{-2}{-3} = \frac{2}{3}$

$M_r \times M_t = -1 \quad \therefore M_t = \frac{-3}{2}$

$y - b = M(x - a) \qquad M = \frac{-3}{2}, \quad (a, b) = (2, 3)$

$y - 3 = \frac{-3}{2}(x - 2)$

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2)

	P		Q		R	
		2	:	1		
x:	−1	→ −4	3	2 ←	5	$\frac{6}{3} = 2$
y:	−1	→ 2	1	1 ←	2	$\frac{3}{3} = 1$
z:	0	→ −2	−2	1 ←	−3	$\frac{3}{3} = 1$

$\therefore \quad Q(3, 1, -2)$

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3) a) i) $f(g(x)) = f(2x) = \sin 2x^\circ$

ii) $g(f(x)) = g(\sin x) = 2\sin x^\circ$

b) $2\sin 2x = 2\sin x$

$2\sin 2x - 2\sin x = 0$

$4\sin x \cos x - 2\sin x = 0$

$2\sin x(2\cos x - 1) = 0$

$2\sin x = 0 \qquad \cos x = 1/2$

$x = 0, 180^\circ, 360^\circ \qquad x = 60^\circ, 300^\circ$

$\therefore \quad x = 0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ$

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4) $M = \tan 45° = 1$.

$M = \frac{dy}{dx} = 4x - 7 = 1$

$\therefore x = 8/4 = 2$

$y = 2\times 2^2 - 7\times 2 + 10$

$= 8 - 14 + 10 = 4$

$\therefore (2, 4)$

5) $AC = \sqrt{2}$ $\quad CB = \sqrt{10}$

$\sin a = \frac{1}{\sqrt{2}}$, $\cos a = \frac{1}{\sqrt{2}}$, $\sin b = \frac{1}{\sqrt{10}}$, $\cos b = \frac{3}{\sqrt{10}}$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

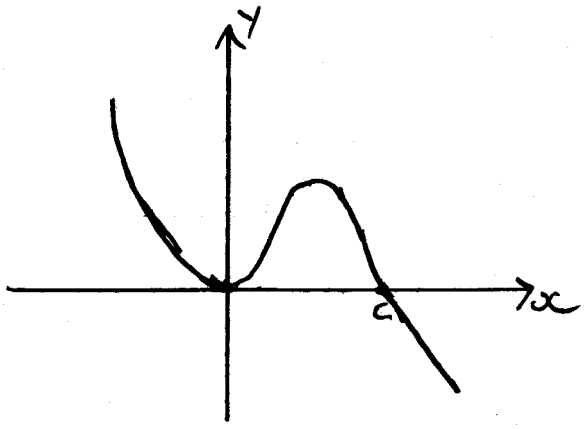
$= \frac{1}{\sqrt{2}} \times \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{10}}$

$= \frac{1}{\sqrt{2}} \left(\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} \right)$

$= \frac{1}{\sqrt{2}} \times \frac{4}{\sqrt{10}} = \frac{4}{\sqrt{2}\sqrt{10}} = \frac{4}{\sqrt{20}}$

$= \frac{4}{\sqrt{4}\sqrt{5}} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$

6)



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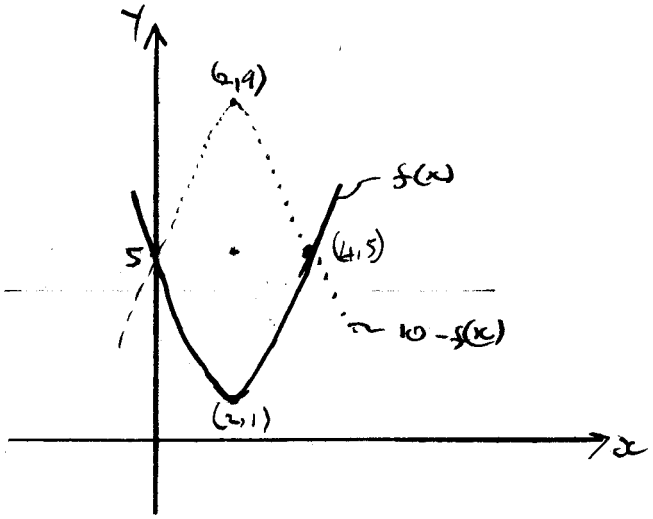
7) a) $f(x) = x^2 - 4x + 5 \quad = (x^2 - 4x + 4) + 5 - 4$

$= \underline{(x-2)^2 + 1}$

b)



c) $10 - (x-2)^2 - 1 = 0$

$(x-2)^2 = 9$

$x - 2 = \pm 9$

$x = -7, 11$

$\therefore \quad \underline{-7 \leq x \leq 11}$

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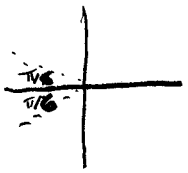
8) a) $y = 2\cos 2x$

b) $2\cos 2x = -\sqrt{3}$

$\cos 2x = \frac{-\sqrt{3}}{2}$

$R.A = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$



$\therefore \quad 2x = \frac{5\pi}{6} \text{ or } \frac{7\pi}{6}$

$\div 2$

$x = \frac{5\pi}{12} \text{ or } \frac{7\pi}{12}$

$\therefore \quad \underline{B\left(\frac{7\pi}{12}, -\sqrt{3}\right)}$

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9) a) $\sin x - \cos x = k\sin(x-a)$

$= K(\sin x\cos a - \cos x\sin a)$

$= K\sin x\cos a - K\cos x\sin a$

$\therefore K\cos a = 1 \qquad K\sin a = 1$

Square and add: $K^2(\cos^2 a + \sin^2 a) = 2$

$K = \sqrt{2}$

$$\frac{K\sin a}{K\cos a} = \frac{1}{1} = \tan a = 1$$

$\therefore \tan a = 1$ sin is +Ve, cos is +Ve

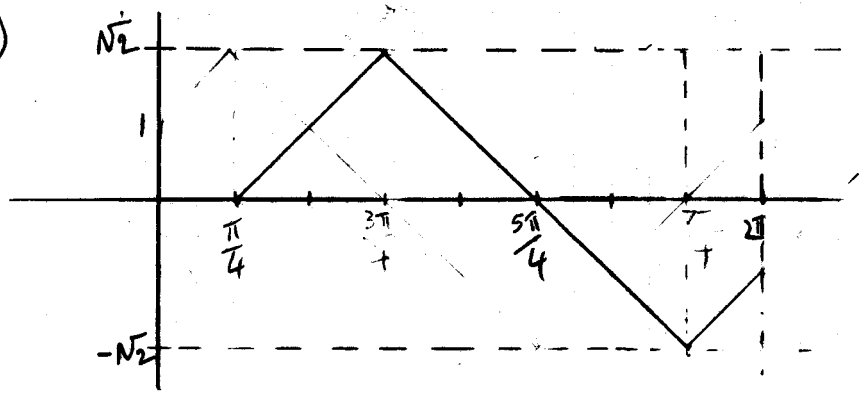
$\therefore$ 1ST Quadrant

$\therefore a = \tan^{-1}(1) = \pi/4$

$\therefore \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$

$\therefore K = \sqrt{2} \qquad a = \frac{\pi}{4}$

b)



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b) a) $f(x) = (8-x^3)^{1/2}$

$f'(x) = \frac{1}{2}(8-x^3)^{-1/2} \times -3x^2$

$= \frac{-3x^2}{2\sqrt{(8-x^3)}}$

b) $\int \frac{x^2}{(8-x^3)^{1/2}} = \frac{-2}{3}(8-x^3)^{1/2} + C$

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11) If $y = kx^n$ then $n = M$, $c = \log_5 k$

$M = \frac{-1}{0.5} = -2$

$\therefore n = -2$

$c = \log_5 k$

$1 = \log_5 k$

$\log_5 5 = \log_5 k$

$\therefore k = 5$

$\therefore y = 5x^{-2}$

2002 – Higher Paper II

1) a) Mid Point AB = $(-2, 2)$

$M_P = \frac{2-2}{5--2} = \frac{0}{7} = 0$

$\therefore$ P is $\underline{y = 2}$

b) $M_{BC} = \frac{2--2}{5--3} = \frac{4}{8} = \frac{1}{2}$ $\quad \therefore M_1 M_2 = -1 \quad \therefore M_2 = -2$

Mid Point BC = $(1, 0)$

$y - b = M(x - a)$ $\quad M = -2$, $\quad (a, b) = (1, 0)$

$y - 0 = -2(x - 1)$

$y = -2x + 2$ $\quad \therefore \underline{y = 2 - 2x}$

c) $y = 2$ $\quad \therefore \quad 2 = 2 - 2x$

$-x = -1$

$x = 0$

$\therefore \underline{(0, 2)}$

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2) a) B(6, 6, 0)

b) $\vec{DA} = a - d = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix} = \underline{\begin{pmatrix} 3 \\ -3 \\ -8 \end{pmatrix}}$ $\quad \vec{DB} = b - d = \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix} = \underline{\begin{pmatrix} 3 \\ 3 \\ -8 \end{pmatrix}}$

c) $\cos\phi = \frac{a \cdot b}{|a||b|} = \frac{\vec{DA} \cdot \vec{DB}}{|\vec{DA}||\vec{DB}|}$

$\vec{DA} \cdot \vec{DB} = (3 \times 3) + (-3 \times 3) + (-8 \times -8) = 9 - 9 + 64 = \underline{64}$

$|\vec{DA}| = \sqrt{3^2 + -3^2 + -8^2} = \underline{9.055}$ $\quad |\vec{DB}| = \sqrt{3^2 + 3^2 - 8^2} = \underline{9.055}$

$\therefore \cos\phi = \frac{64}{9.055 \times 9.055} = 0.781$

$\therefore \phi = \cos^{-1}(0.781) = \underline{38.6^\circ}$

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3) a) Max at $\frac{dy}{dx} = 0$: $\frac{dy}{dx} = 6x^2 - 14x + 4 = 0$

$2(3x^2 - 7x + 2) = 0$

$2(3x - 1)(x - 2) = 0$

$\therefore x = \frac{1}{3}$ OR $x = 2$

$\therefore$ at Max $x = \frac{1}{3}$

b)

	2	-7	4	4
2		4	-6	-4
	2	-3	-2	0

$\therefore (x-2)(2x^2 - 3x - 2)$

$(x-2)(2x+1)(x-2)$

c) $A_x = -\frac{1}{2}$, $A_y = 0$

$\therefore A(-\frac{1}{2}, 0)$

$\therefore x < -\frac{1}{2}$

4) a) $u_{n+1} = au_n + b$

$u_{n+1} = 0.8u_n + 0.5$ $\quad -1 < a < 1$ $\therefore$ limit exists

Limit $L = \frac{b}{1-a}$

$L = \frac{0.5}{1-0.8} = \frac{0.5}{0.2} = 2.5\text{m}$

b) $L = 2$ find a

$\therefore 2 = \frac{0.5}{(1-a)} = 2(1-a) = 0.5$

$2 - 2a = 0.5$

$-2a = -1.5$

$a = 0.75$

$\therefore$ Trim 25%

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5) Find limits: $1+10x-2x^2 = 1+5x-x^2$

$$x^2 - 5x = 0$$

$$x(x-5) = 0$$

$$\underline{x=0} \quad , \underline{x=5}$$

$$\therefore \text{Area} = \int_0^5 (1+10x-2x^2) - (1+5x-x^2)\,dx$$

$$= \int_0^5 (5x - x^2)\,dx = \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$$

$$= \left(\frac{5\times 5^2}{2} - \frac{5^3}{3}\right) - 0 = 62.5 - 41.7$$

$$= \underline{20.8 \text{ units}^2}$$

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6)

$$M = \frac{dy}{dx} = 2\cos\left(x - \frac{\pi}{6}\right) \text{ at } \frac{\pi}{3}$$

$$M = 2\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3}}{2} = \underline{\sqrt{3}}$$

at $x = \frac{\pi}{3}$: $y = 2\sin\left(\frac{\pi}{6}\right) = 2\times\frac{1}{2} = \underline{1}$

$y - b = M(x-a)$ $\quad M = \sqrt{3}$ $\quad (a,b) = (\pi/3, 1)$

$$y - 1 = \sqrt{3}\left(x - \frac{\pi}{3}\right)$$

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7) Cuts x-axis at $y=0$

$$\therefore \log_3(x-2) + 1 = 0$$

$$\log_3(x-2) = -1$$

$$-1 = \log_3 \frac{1}{3}$$

$$\log_3(x-2) = \log_3 \tfrac{1}{3}$$

$$x - 2 = \tfrac{1}{3}$$

$$x = 2\tfrac{1}{3}$$

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8) If $a = \frac{dv}{dt}$ then $v = \int a\,dt$

$$v = \int 2(4-t)^{1/2}\,dt = \frac{2}{-1 \times \frac{3}{2}}(4-t)^{-1/2} + C$$

$$= -\frac{4}{3}(4-t)^{3/2} + C$$

$$= \frac{-4\sqrt{(4-t)^3}}{3} + C$$

at $t = 0$, $v = 0$

$$\therefore \quad 0 = \frac{-4\sqrt{(4)^3}}{3} + C$$

$$0 = -10.67 + C$$

$$\therefore C = 10.67$$

$$\therefore \quad v = \frac{-4\sqrt{(4-t)^3}}{3} + 10.67$$

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9) $b^2 - 4ac \geq 0$

$a = (1-2k) \qquad b = -5k \qquad c = -2k$

$= (-5k)^2 - (4 \times (1-2k) \times -2k)$

$= 25k^2 - (-8k(1-2k))$

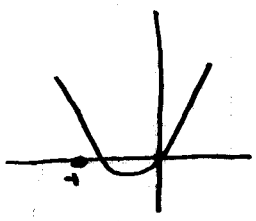
$= 25k^2 - (-2k + 16k^2)$

$= 25k^2 + 8k - 16k^2$

$= \underline{9k^2 + 8k \geq 0}$

$\therefore k(9k+8) \geq 0$

$k = 0 \quad$ or $\quad k = \frac{-8}{9}$



$\therefore$ all integer values are greater than or equal to zero.

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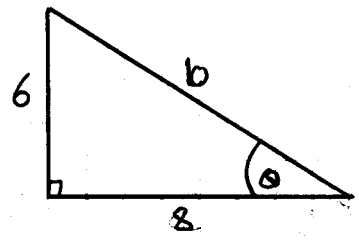
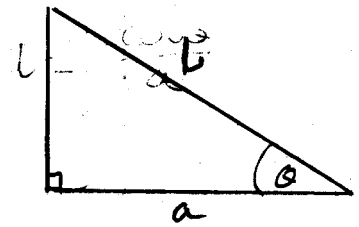
10) a) i) Similar triangles.



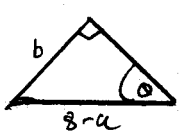
$\therefore \sin\theta = \frac{6}{10} = \frac{3}{5} \quad \cos\theta = \frac{8}{10} = \frac{4}{5}, \quad \tan\theta = \frac{6}{8} = \frac{3}{4}$



$\cos\theta = \frac{a}{L} \quad \therefore L = \frac{a}{\cos\theta} = \frac{a}{\frac{4}{5}}$

$\therefore \underline{L = \frac{5a}{4}}$

ii)



$\sin\theta = \frac{b}{8-a} \quad \therefore b = \frac{3}{5}(8-a).$

$\therefore A = \frac{5}{4}a \times \frac{3}{5}(8-a) = \underline{\frac{3}{4}a(8-a)}$

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10) b) Max at $\frac{dA}{da} = 0$

$\therefore \quad A = 6a - \frac{3}{4}a^2$

$\frac{dA}{da} = 6 - \frac{3}{2}a = 0$

$\therefore \quad \frac{3}{2}a = 6$

$a = \frac{2 \times 6}{3} = \underline{4}$

Check:

a	4^-	4	4^+
$\frac{dA}{da}$	$+$	0	$-$
SHAPE	/	—	\

$\therefore$ Max at $a = 4$