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Heinemann

Higher Maths Text Book

Worked Solutions

Ex 13S
Applications

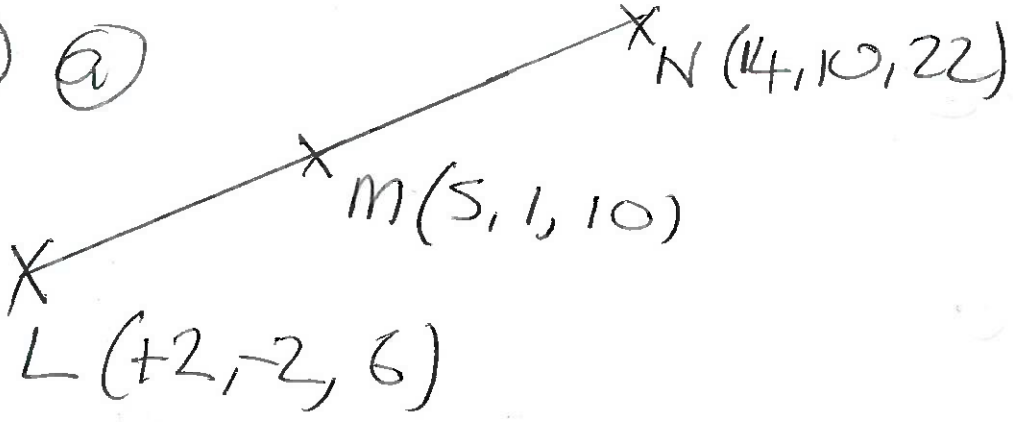
Scalar product $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

or $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Ex 13S Questions 1ab, 2abc, 4ab, 5abc, 6ab

Worked solutions courtesy of Mr R Milton

(1) (a)



$\overrightarrow{LM} = m - l$

$= \begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$\overrightarrow{MN} = n - m$

$= \begin{bmatrix} 14 \\ 10 \\ 22 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

$= 3\overrightarrow{LM}$

$\therefore \overrightarrow{LM} = 3\overrightarrow{MN}$ ✓

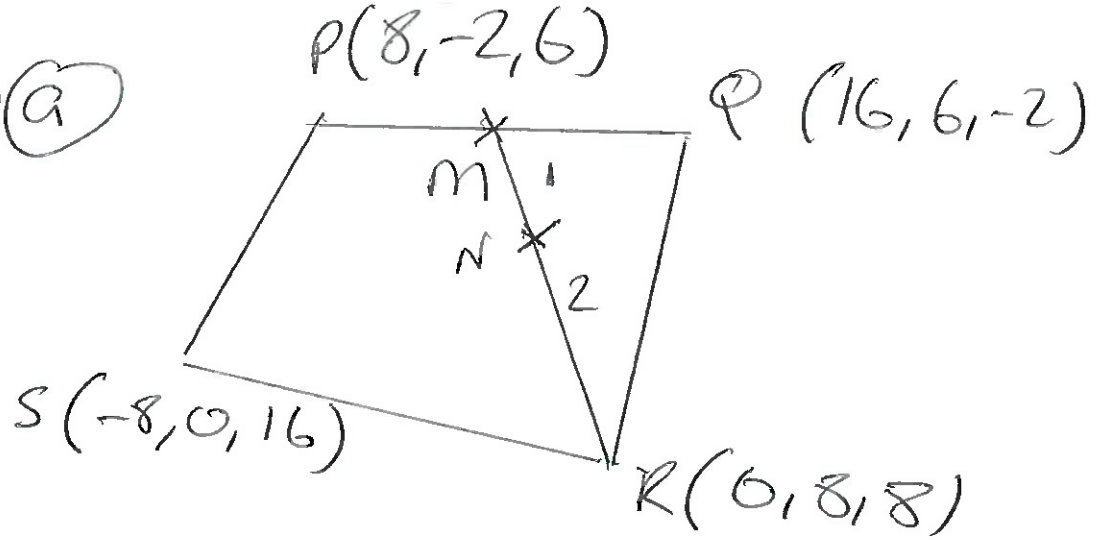
$\Rightarrow$ LM AND MN ARE PARALLEL WITH A COMMON POINT M ✓

$\Rightarrow$ L, M, N ARE COLLINEAR

(b)

RATIO 3 : 1 ✓

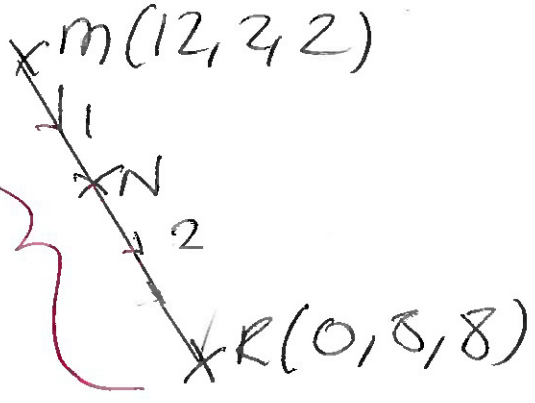
(2) (a)



$$m\left(\frac{8+16}{2}, \frac{-2+6}{2}, \frac{6-2}{2}\right)$$

$$\Rightarrow m(12,2,2) \checkmark$$

(b)



$$2\vec{mN} = 1\vec{NR}$$

$$2(n-m) = r-n$$

$$2n-2m = r-n$$

$$\Rightarrow 3n = r+2m$$

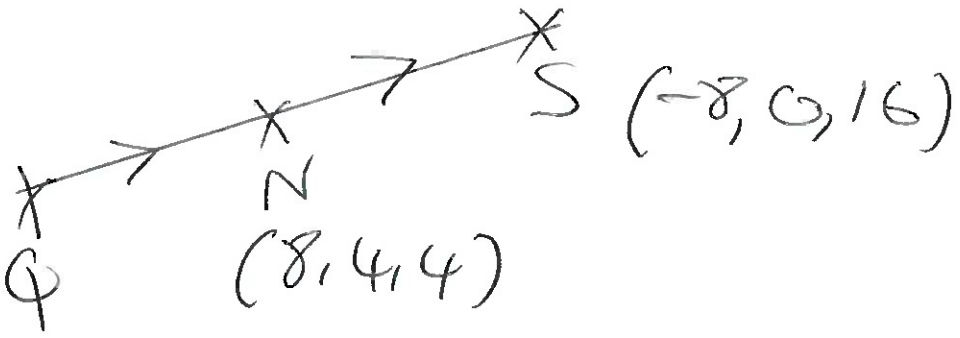
$$n = \frac{1}{3}[r+2m]$$

$$= \frac{1}{3}\left[\begin{bmatrix}0\\8\\8\end{bmatrix}+\begin{bmatrix}24\\4\\4\end{bmatrix}\right]$$

$$= \frac{1}{3}\begin{bmatrix}24\\12\\12\end{bmatrix} = \begin{bmatrix}8\\4\\4\end{bmatrix}$$

$$\Rightarrow N(8,4,4) \checkmark$$

(c)



$$\vec{QN} = n - q = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 16 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ -2 \\ 6 \end{pmatrix}$$

$$\vec{NS} = s - n = \begin{pmatrix} -8 \\ 0 \\ 16 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} -16 \\ -4 \\ 12 \end{pmatrix}$$

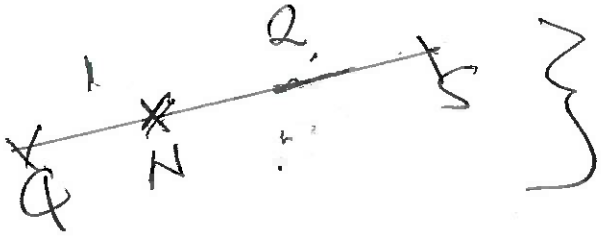
$$= 2 \begin{pmatrix} -8 \\ -2 \\ 6 \end{pmatrix}$$

$$= 2\vec{QN}$$

SINCE $\vec{NS} = 2\vec{QN}$

AND N IS A COMMON POINT THEN ✓

Q, N, S ARE COLLINEAR

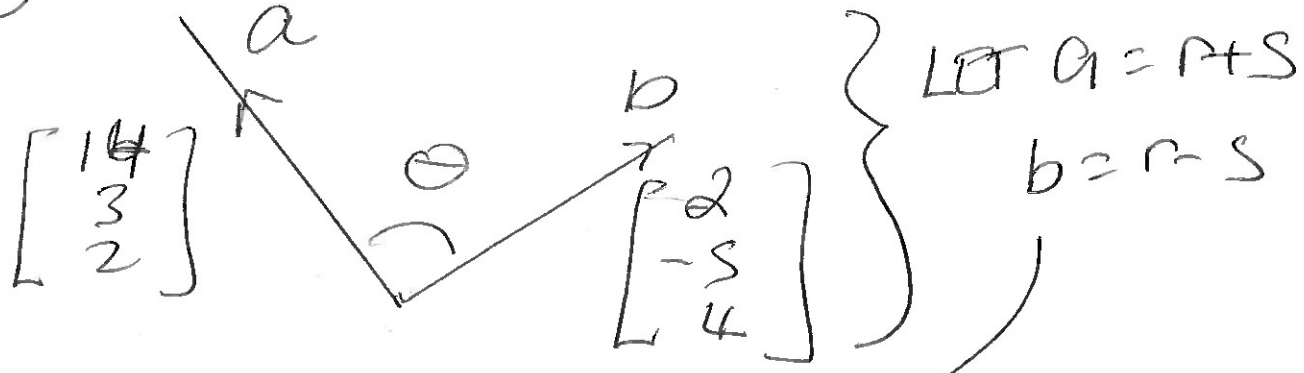


} RATIO 1:2 ✓

4 (a) $r = \begin{bmatrix} 8 \\ -1 \\ 3 \end{bmatrix}$ $s = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}$

$r+s = \begin{bmatrix} 14 \\ 3 \\ 2 \end{bmatrix}$ ✓ $r-s = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$ ✓

(b)



$\cos\theta = \dfrac{a \cdot b}{|a||b|}$

$a \cdot b = (14 \times 2) + (3 \times -5) + (2 \times 4)$

$= 28 - 15 + 8 = 21$

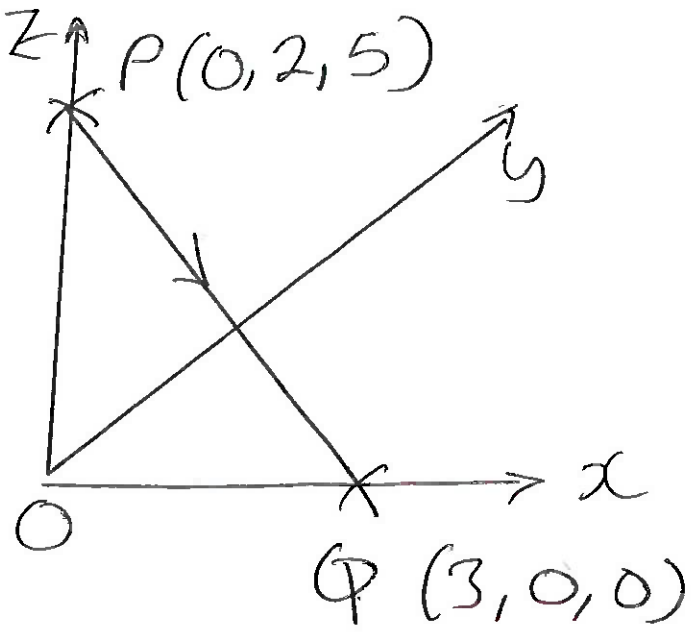
$|a| = \sqrt{14^2 + 3^2 + 2^2} = \sqrt{209}$

$|b| = \sqrt{2^2 + (-5)^2 + 4^2} = \sqrt{45}$

$\cos\theta = \dfrac{21}{\sqrt{209}\sqrt{45}} \Rightarrow \theta = \cos^{-1}(0.2165)$

$= 77.5°$ ✓

(5)



(a) $\overrightarrow{PQ} = q - p = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$ ✓

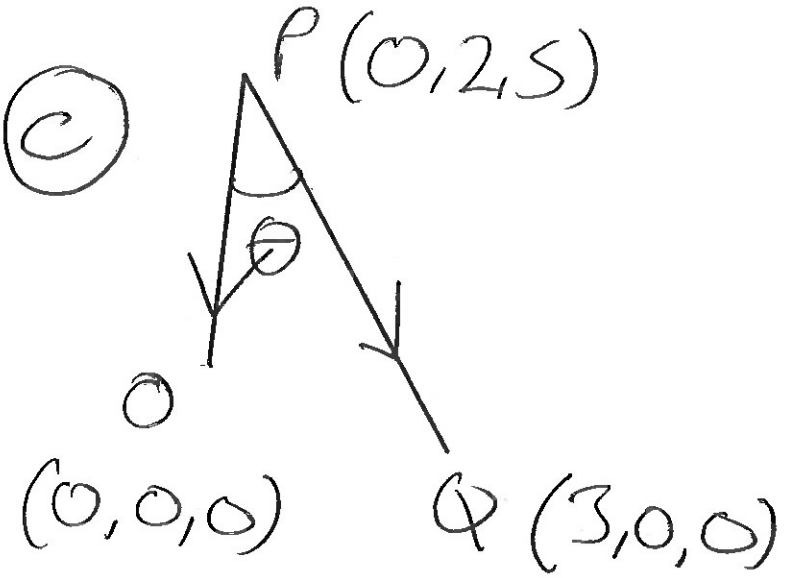
$\overrightarrow{PQ} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$ ✓

(b) $|\overrightarrow{PQ}| = \sqrt{3^2 + (-2)^2 + (-5)^2}$

$= \sqrt{9 + 4 + 25}$

$= \sqrt{38}$ ✓

(c)



$$\cos\theta = \frac{a.b}{|a||b|}$$

$$\cos\theta = \frac{\vec{PO}.\vec{PQ}}{|PO|\,|PQ|}$$

$$\vec{PO} = O - P$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \\ -5 \end{bmatrix}$$

$$\vec{PQ} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$$

$$|\vec{PO}| = \sqrt{0 + 4 + 25} = \underline{\sqrt{29}}$$

$$\vec{PO}.\vec{PQ} = (0\times3) + (-2\times-2) + (-5\times-5)$$

$$= 4 + 25$$

$$= \underline{29}$$

$$\cos\theta = \frac{29}{\sqrt{29} \times \sqrt{38}}$$

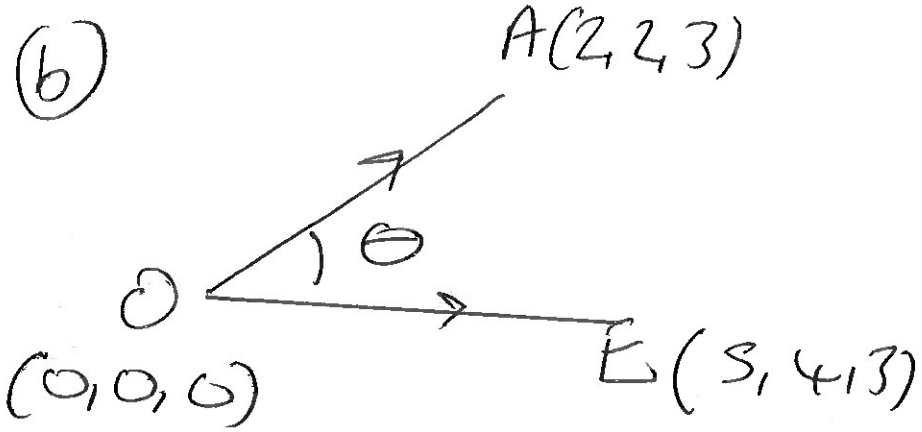
$$\theta = \cos^{-1}\left[\frac{29}{\sqrt{29}\sqrt{38}}\right]$$

$$= 29.1^{\circ}$$ ✓

(6) (a) $E\left[\frac{8+2}{2}, \frac{2+6}{2}, \frac{3+3}{2}\right]$

$E[5, 4, 3]$ ← HALF WAY BETWEEN B & D

(b)



$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{\vec{OA} \cdot \vec{OE}}{|OA| \times |OE|}$$

$$\vec{OA} = a - O = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{OE} = e - O = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

$$\vec{OA} \cdot \vec{OE} = 10 + 8 + 9 = \underline{27}$$

$$|\vec{OA}| = \sqrt{4+4+9}$$

$$= \sqrt{17}$$

$$|\vec{OE}| = \sqrt{25+16+9}$$

$$= \sqrt{50}$$

$$\cos\theta = \frac{27}{\sqrt{17}\sqrt{50}}$$

$$\theta = \cos^{-1}\left[\frac{27}{\sqrt{17}\sqrt{50}}\right]$$

$$= \cos^{-1}[0.92609]$$

$$= 22.2^\circ$$ ✓