Recurrence Relations

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

or $\mathbf{a.b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Table of standard integrals:

$f(x)$	$\int f(x)dx$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\cos ax$	$\frac{1}{a}\sin ax + c$

Recurrence Relations

2019 P2 Q4	In a forest, the population of a species of mouse is falling by 2·7% each year. To increase the population scientists plan to release 30 mice into the forest at the end of March each year. (a) u_n is the estimated population of mice at the start of April, n years after the population was first estimated. It is known that u_n and u_{n+1} satisfy the recurrence relation $u_{n+1} = au_n + b$. State the values of a and b. The scientists continue to release this species of mouse each year. (b) (i) Explain why the estimated population of mice will stabilise in the long term. (ii) Calculate the long term population to the nearest hundred.	1 1 2
2019 P1 Q4	A sequence is generated by the recurrence relation $u_{n+1} = mu_n + c$, where the first three terms of the sequence are 6, 9 and 11. (a) Find the values of m and c. (b) Hence, calculate the fourth term of the sequence.	3 1
2017 P2 Q8	Sequences may be generated by recurrence relations of the form $u_{n+1} = k\,u_n - 20$, $u_0 = 5$ where $k \in \mathbb{R}$. (a) Show that $u_2 = 5k^2 - 20k - 20$. (b) Determine the range of values of k for which $u_2 < u_0$.	2 4
2017 P1 Q9	A sequence is generated by the recurrence relation $u_{n+1} = m\,u_n + 6$ where m is a constant. (a) Given $u_1 = 28$ and $u_2 = 13$, find the value of m. (b) (i) Explain why this sequence approaches a limit as $n \to \infty$. (ii) Calculate this limit.	2 1 2

2016 P1 Q3

A sequence is defined by the recurrence relation $u_{n+1} = \frac{1}{3}u_n + 10$ with $u_3 = 6$.

(a) Find the value of u_4. 1

(b) Explain why this sequence approaches a limit as $n \to \infty$. 1

(c) Calculate this limit. 2

2015 P2 Q3

Mark

A version of the following problem first appeared in print in the 16th Century.

A frog and a toad fall to the bottom of a well that is 50 feet deep.
Each day, the frog climbs 32 feet and then rests overnight. During the night, it slides down $\frac{2}{3}$ of its height above the floor of the well.
The toad climbs 13 feet each day before resting.

Overnight, it slides down $\frac{1}{4}$ of its height above the floor of the well.

Their progress can be modelled by the recurrence relations:

- $f_{n+1} = \frac{1}{3}f_n + 32, \quad f_1 = 32$
- $t_{n+1} = \frac{3}{4}t_n + 13, \quad t_1 = 13$

where f_n and t_n are the heights reached by the frog and the toad at the end of the nth day after falling in.

(*a*) Calculate t_2, the height of the toad at the end of the second day. 1

(*b*) Determine whether or not either of them will eventually escape from the well. 5

2013 P2 Q1

The first three terms of a sequence are 4, 7 and 16.

The sequence is generated by the recurrence relation

$$u_{n+1} = mu_n + c, \text{ with } u_1 = 4.$$

Find the values of m and c. 4

2011 P2 Q3

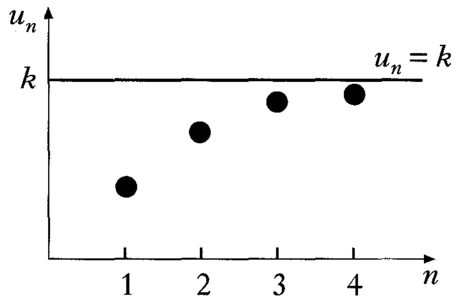
(*a*) A sequence is defined by $u_{n+1} = -\frac{1}{2}u_n$ with $u_0 = -16$. 1
Write down the values of u_1 and u_2.

(*b*) A second sequence is given by 4, 5, 7, 11, ...
It is generated by the recurrence relation $v_{n+1} = pv_n + q$ with $v_1 = 4$. 3
Find the values of p and q.

(*c*) Either the sequence in (*a*) or the sequence in (*b*) has a limit.

(i) Calculate this limit. 3

(ii) Why does the other sequence not have a limit?

2007 P1	**7.** A sequence is defined by the recurrence relation $$u_{n+1} = \frac{1}{4}u_n + 16,\ u_0 = 0.$$ (*a*) Calculate the values of u_1, u_2 and u_3. Four terms of this sequence, u_1, u_2, u_3 and u_4 are plotted as shown in the graph. As $n \to \infty$, the points on the graph approach the line $u_n = k$, where k is the limit of this sequence.  (*b*) (i) Give a reason why this sequence has a limit. (ii) Find the exact value of k.	3 3
2006 P1	**4.** A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}8u_n + 12$, $u_0 = 4$. (*a*) State why this sequence has a limit. (*b*) Find this limit.	1 2
2005 P1	**6.** (*a*) The terms of a sequence satisfy $u_{n+1} = ku_n + 5$. Find the value of k which produces a sequence with a limit of 4. (*b*) A sequence satisfies the recurrence relation $u_{n+1} = mu_n + 5$, $u_0 = 3$. (i) Express u_1 and u_2 in terms of m. (ii) Given that $u_2 = 7$, find the value of m which produces a sequence with no limit.	2 5
2004 P2	**4.** A sequence is defined by the recurrence relation $u_{n+1} = ku_n + 3$. (*a*) Write down the condition on k for this sequence to have a limit. (*b*) The sequence tends to a limit of 5 as $n \to \infty$. Determine the value of k.	1 3
2003 P1	**4.** A recurrence relation is defined by $u_{n+1} = pu_n + q$, where $-1 < p < 1$ and $u_0 = 12$. (*a*) If $u_1 = 15$ and $u_2 = 16$, find the values of p and q. (*b*) Find the limit of this recurrence relation as $n \to \infty$.	2 2
2002W P2	**3.** (*a*) Calculate the limit as n $\to \infty$ of the sequence defined by $u_{n+1} = 0{\cdot}9u_n + 10$, $u_0 = 1$. (*b*) Determine the least value of n for which u_n is greater than half of this limit and the corresponding value of u_n.	3 2

2002 P2	**4.** A man decides to plant a number of fast-growing trees as a boundary between his property and the property of his next door neighbour. He has been warned, however, by the local garden centre that, during any year, the trees are expected to increase in height by 0·5 metres. In response to this warning he decides to trim 20% off the height of the trees at the start of any year. (*a*) If he adopts the "20% pruning policy", to what height will he expect the trees to grow in the long run? (*b*) His neighbour is concerned that the trees are growing at an alarming rate and wants assurances that the trees will grow no taller than 2 metres. What is the minimum percentage that the trees will need to be trimmed each year so as to meet this condition?	3 3
2001 P2	**3.** On the first day of March, a bank loans a man £2500 at a fixed rate of interest of 1·5% per month. This interest is added on the last day of each month and is calculated on the amount due on the first day of the month. He agrees to make repayments on the first day of each subsequent month. Each repayment is £300 except for the smaller final amount which will pay off the loan. (*a*) The amount that he owes at the start of each month is taken to be the amount still owing just after the monthly repayment has been made. Let u_n and u_{n+1} represent the amounts that he owes at the starts of two successive months. Write down a recurrence relation involving u_{n+1} and u_n. (*b*) Find the date and the amount of the final payment.	2 4
Specimen 2 P2	**4.** Two sequences are defined by the recurrence relations $u_{n+1} = 0{\cdot}2u_n + p, \quad u_0 = 1$ and $v_{n+1} = 0{\cdot}6v_n + q, \quad v_0 = 1.$ (*a*) Explain why each of these sequences has a limit. (*b*) If both sequences have the same limit, express p in terms of q.	3
Specimen P2 Q2	A wildlife reserve has introduced conservation measures to build up the population of an endangered mammal. Initially the reserve population of the mammal was 2000. By the end of the first year there were 2500 and by the end of the second year there were 2980. It is believed that the population can be modelled by the recurrence relation: $u_{n+1} = au_n + b$, where a and b are constants and n is the number of years since the reserve was set up. (a) Use the information above to find the values of a and b. (b) Conservation measures will end if the population stabilises at over 13 000. Will this happen? Justify your answer.	4 3

Specimen 1 P1 Q1	A sequence is defined by the recurrence relation $u_{n+1} = 0 \cdot 3u_n + 5$ with first term u_1. (*a*) Explain why this sequence has a limit as *n* tends to infinity. (*b*) Find the **exact** value of this limit.	1 2
Specimen 1 P2	**2.** Trees are sprayed weekly with the pesticide, "Killpest", whose manufacturers claim it will destroy 60% of all pests. Between the weekly sprayings, it is estimated that 300 new pests invade the trees. A new pesticide, "Pestkill", comes onto the market. The manufacturers claim that it will destroy 80% of existing pests but it is estimated that 360 new pests per week will invade the trees. Which pesticide will be more effective in the long term?	5